Case 2 – $\sigma_1^2 \neq \sigma_2^2$ and $n_1 = n_2$

The random variable $t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)_0}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$ is $t$ with $n_1 + n_2 - 2$ degrees of freedom.

The procedure is best seen via an example. Suppose we have the hypothesis that the means are equal, i.e., that $(\mu_1 - \mu_2)_0 = 0$, and suppose we have the samples $\{1, 2, 3, 4, 5\}$ and $\{-3, -1, 5, 7, 12\}$. The procedure is as follows:

(1) State the null hypothesis, the alternative hypothesis, and the significance level:

$H_0: \mu_1 - \mu_2 = 0$

$H_A: \mu_1 - \mu_2 \neq 0$

$\alpha = .05$

(2) Determine the critical region

Given $H_A: \mu_1 - \mu_2 \neq 0$, the critical region is $CR = \{t \mid t < -t_{\alpha/2}\} \cup \{t \mid t > t_{\alpha/2}\}$

For our example $n_1 = 5$ and $n_2 = 5$, so the $t$ variable has $5+5-2 = 8$ degrees of freedom, and the critical region is $\{t \mid t < -2.3060\} \cup \{t \mid t > 2.3060\}$.

Graphically, the critical region is as follows:

(3) Determine the value of the test statistic

The samples $\{1, 2, 3, 4, 5\}$ and $\{-3, -1, 5, 7, 12\}$ yield

$\bar{x}_1 = \frac{1+2+3+4+5}{5} = \frac{15}{5} = 3$, 

$\bar{x}_2 = \frac{-3-1+5+7+12}{5} = \frac{22}{5} = 4.4$.
\[ s_1^2 = \frac{(1-3)^2 + (2-3)^2 + (3-3)^2 + (4-3)^2 + (5-3)^2}{5-1} = \frac{10}{4} = 2.5, \]
\[ \bar{x}_2 = \frac{-3+(-1)+5+7+12}{5} = \frac{20}{5} = 4, \text{ and} \]
\[ s_2^2 = \frac{(-3-4)^2 + (-1-4)^2 + (5-4)^2 + (7-4)^2 + (12-4)^2}{5-1} = \frac{148}{4} = 37. \]

Therefore, the value of the test statistic is
\[ t^* = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{(3-4)-(0)}{\sqrt{\frac{2.5}{5} + \frac{37}{5}}} = -0.355784. \]

(4) Locate the value of the test statistics relative to the critical region
Since \(-2.3060 < -0.355784 < 2.3060\), \(t^*\) is not in the critical region.

Graphically, we have the following:

(5) Make the decision
Since \(t^*\) is not in the critical region, we maintain the null hypothesis
\(H_0: \mu_1 - \mu_2 = 0.\)

Again, note that maintaining \(H_0\) is not the same thing as accepting \(H_0\) as true. All we have done in maintaining \(H_0\) is to conclude that we cannot (yet) falsify \(H_0.\) We do not know that \(H_0\) is true; we merely know that we have failed to falsify \(H_0.\)
Now suppose that we suspect that $\mu_1$ is, in fact, smaller than $\mu_2$. We address this suspicion via the following hypotheses:

$$H_0: \mu_1 - \mu_2 = 0$$
$$H_A: \mu_1 - \mu_2 < 0$$

Note that the test statistic does not change (since $H_0$ did not change), but the critical region does change (since $H_A$ changed). The critical region is now $\text{CR} = \{ t \mid t < -t_{\alpha/2} \}$, so that we have $\text{CR} = \{ t \mid t < -1.8595 \}$. Graphically, we have the following:

![Graph showing critical region with $t^* = -0.355784$ and threshold at $-2.8595$.]

Again, we maintain the null hypothesis $H_0: \mu_1 - \mu_2 = 0$ and we abandon our suspicion that $\mu_1$ is smaller than $\mu_2$.

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The procedure is best seen via an example. Suppose we have the hypothesis that the means are equal, i.e., that $(\mu_1 - \mu_2)_0 = 0$, and suppose we have the samples \{1, 2, 3, 4, 5\} and \{-3, -1, 5, 7, 12\}. The procedure is as follows:

1. State the null hypothesis, the alternative hypothesis, and the significance level:
   \[ H_0: \mu_1 - \mu_2 = 0 \]
   \[ H_A: \mu_1 - \mu_2 \neq 0 \]
   \[ \alpha = .05 \]

2. Determine the critical region
   Given $H_A: \mu_1 - \mu_2 \neq 0$, the critical region is $CR = \{t \mid t < -t_{\alpha/2} \cup \{t \mid t > t_{\alpha/2}\}$

   For our example $n_1 = 5$ and $n_2 = 5$, so the $t$ variable has $5+5-2 = 8$ degrees of freedom, and the critical region is $\{t \mid t < -2.3060 \cup \{t \mid t > 2.3060\}$.

   Graphically, the critical region is as follows:

3. Determine the value of the test statistic
   The samples \{1, 2, 3, 4, 5\} and \{-3, -1, 5, 7, 12\} yield
   \[ \bar{x}_1 = \frac{1+2+3+4+5}{5} = \frac{15}{5} = 3, \]
   \[ s_1^2 = \frac{(1-3)^2 + (2-3)^2 + (3-3)^2 + (4-3)^2 + (5-3)^2}{5-1} = \frac{10}{4} = 2.5, \]
   \[ \bar{x}_2 = \frac{-3+1+5+7+12}{5} = \frac{20}{5} = 4, \text{ and} \]
   \[ s_2^2 = \frac{(-3-4)^2 + (-1-4)^2 + (5-4)^2 + (7-4)^2 + (12-4)^2}{5-1} = \frac{148}{4} = 37. \]

   \[ s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} = \frac{(5-1)2.5 + (5-1)37}{5+5-2} = \frac{158}{8} = 19.75. \]

   Therefore, the value of the test statistic is
\[ t^* = \frac{(x_1 - x_2) - (\mu_1 - \mu_2)_0}{\sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}}} = \frac{(3 - 4) - (0)}{\sqrt{\frac{19.75}{5} + \frac{19.75}{5}}} = -0.355784. \]

(4) Locate the value of the test statistics relative to the critical region

Since \(-2.3060 < -0.355784 < 2.3060\), \(t^*\) is not in the critical region.

Graphically, we have the following:

(5) Make the decision

Since \(t^*\) is not in the critical region, we maintain the null hypothesis \(H_0: \mu_1 - \mu_2 = 0\).

Again, note that maintaining \(H_0\) is not the same thing as accepting \(H_0\) as true. All we have done in maintaining \(H_0\) is to conclude that we cannot (yet) falsify \(H_0\). We do not know that \(H_0\) is true; we merely know that we have failed to falsify \(H_0\).
Now suppose that we suspect that $\mu_1$ is, in fact, smaller than $\mu_2$. We address this suspicion via the following hypotheses:

\[ H_0: \mu_1 - \mu_2 = 0 \]
\[ H_A: \mu_1 - \mu_2 < 0 \]

Note that the test statistic does not change (since $H_0$ did not change), but the critical region does change (since $H_A$ changed). The critical region is now $CR = \{t \mid t < -t_{\alpha/2}\}$, so that we have $CR = \{t \mid t < -1.8595\}$. Graphically, we have the following:

![Graph of t distribution with critical region]

Again, we maintain the null hypothesis $H_0: \mu_1 - \mu_2 = 0$ and we abandon our suspicion that $\mu_1$ is smaller than $\mu_2$.

Some will respond with a statement to the effect that “But the sample averages of 3 and 4 make it clear that $\mu_1$ is smaller than $\mu_2$”. This is simply weak intuition. The hypothesis test reveals that, given our data, we cannot discern that $\mu_1$ is smaller than $\mu_2$. If the suspicion is firmly held, then the solution is to conduct experiments that yield very large, and thereby very representative, samples, and to engage in repetitive testing.

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